A SHORT PROOF OF BING'S CHARACTERIZATION OF S^3

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Abstract. We give a short proof of Bing's characterization of S^3: a compact, connected 3-manifold M is S^3 if and only if every knot in M is isotopic into a ball.

Let M be a closed orientable 3-manifold. We assume familiarity with the basic notions of irreducible and prime 3-manifolds (see, *e.g.,* [3] or [4]) and the basic results about Heegaard splittings of compact 3-manifolds (see, *e.g.,* [7]). By *genus* we always mean Heegaard genus. A knot $k \subset M$ (that is, a smooth embedding of the circle into M) is called *irreducible* if its exterior $E(k) = M \setminus N(k)$ is an irreducible 3-manifold. In his own words, Bing's Theorem [1, Theorem 1] is:

Theorem 1 (Bing)**.** *A compact, connected 3-manifold M is topologically S^3 if each simple closed curve in M lies in a topological cube in M.*

By "topological cube" Bing meant what we usually call a ball. Clearly, any knot in S^3 is contained in a ball. If a knot k in a manifold $M \ncong S^3$ is contained in a ball (say B) then by considering the boundary of B we see that k is not irreducible. Thus, Theorem 1 follows from:

Theorem 2. *Any compact, connected 3-manifold admits an irreducible knot.*

In [5, Theorem 8.1] Jaco and Rubinstein gave a very short proof of Theorem 1 for irreducible manifolds, but their proof relies on the existence of 0-efficient triangulations. The purpose of this note is giving a short, elementary proof of Theorem 1.
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1. The proof

We prove Theorem 2; as remarked above Theorem 1 follows.
Case One: M is prime. First, when M has genus at most one, let k be a knot on a Heegaard torus (in M) with $E(k)$ a Seifert fibered space over the disk with 2 exceptional fibers, which is irreducible.

Second, when M has genus two or more, then $M \ncong S^2 \times S^1$ and hence is irreducible. Let $M = V_1 \cup_\Sigma V_2$ be a minimal genus Heegaard splitting of M. By Waldhausen [8] (see also [7, Theorem 3.8]) Σ is irreducible. Let k be a core of a 1-handle in V_1. Then Σ is an irreducible Heegaard surface for $E(k)$; Haken [2] (see also [7, Theorem 3.4]) showed that every Heegaard splitting of a reducible manifold is reducible; hence, $E(k)$ is irreducible.

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Remark 3. In Case One, $\partial E(k)$ is incompressible. For manifolds of genus one or less this is so by construction of k. For manifold of genus two or more, if $\partial E(k)$ compressed then (since $E(k)$ is irreducible) $E(k)$ would be a solid torus; but that implies M has genus at most one, contradiction.

Case Two: M **is composite.** By Kneser [6] M has a prime decomposition as $M \cong M_1 \# \cdots \# M_n$ with M_i prime $(i = 1, \dots, n)$. Let $k_i \subset M_i$ be the knot obtained in Case One, let $k = \#_{i=1}^n k_i \subset M$ be their connected sum, and let $\mathcal{A} \subset E(k)$ be a collection of annuli that decomposes k into its summands, that is, the components of $E(k)$ cut open along $\mathcal{A}$ are homeomorphic to $E(k_i)$ $(i = 1, \dots, n)$.

Let S be a sphere in $E(k)$, we will prove that S bounds a ball. By isotopy of S, minimize $S \cap \mathcal{A}$. Assume that $S \cap \mathcal{A} \neq \emptyset$. Since $\chi(S) = 2$, S cut open along $\mathcal{A}$ has disk components, and let D be such a disk. Then D is contained is some component of $E(k)$ cut open along $\mathcal{A}$ (which is homeomorphic to $E(k_i)$, for some i). By Remark 3, $\partial E(k_i)$ is incompressible and hence D is boundary parallel, contradicting the minimality assumption. Hence $S \cap \mathcal{A} = \emptyset$, and S is contained in a component of $E(k)$ cut open along $\mathcal{A}$. By the construction in Case One S bounds a ball. Thus every sphere in $E(k)$ bounds a ball and k is an irreducible knot, completing the proof of Theorems 2 and 1.

References

[1] R. H. Bing, *Necessary and sufficient conditions that a* 3*-manifold be* S^3, Ann. of Math. (2) **68** (1958), 17–37.

[2] Wolfgang Haken, *Some results on surfaces in* 3*-manifolds*, Studies in Modern Topology, Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.), 1968, pp. 39–98.

[3] John Hempel, 3*-Manifolds*, Princeton University Press, Princeton, N. J., 1976, Ann. of Math. Studies, No. 86.

[4] William Jaco, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics, vol. 43, American Mathematical Society, Providence, R.I., 1980.

[5] William Jaco and J. Hyam Rubinstein, 0*-efficient triangulations of 3-manifolds*, J. Differential Geom. **65** (2003), no. 1, 61–168.

[6] H. Kneser, *Geschlossene flächen in dreidimensionalen Mannigfaltigkeiten*, Jahresbericht der Deut. Math. Verein. **38** (1929), 248–260.

[7] Martin Scharlemann, *Heegaard splittings of 3-manifolds*, Low dimensional topology, New Stud. Adv. Math., vol. 3, Int. Press, Somerville, MA, 2003, pp. 25–39.

[8] Friedhelm Waldhausen, *Heegaard-Zerlegungen der* 3*-Sphäre*, Topology **7** (1968), 195–203.

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